Arbitrary Order Perturbation Theory for Time–Discrete Vlasov Systems with Drift Maps and Poisson Type Collective Kick Maps.

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Abstract

The well established model[1, 2, 3] for studying the microbunching instability driven by longitudinal space charge in ultra-relativistic bunches in FEL-like beamlines can be identified as a time-discrete Vlasov model with general drift maps and Poisson type collective kick maps. This model can in principle be solved exactly using the method of characteristics (Perron-Frobenius operator method). Here we describe a higher order perturbative approach based on the Frechet derivative of the Perron-Frobenius operator, and show that is in prinicple suited to analytically compute approximations to the microbunching gain functions.

References

[1] E.Saldin, E.Schneidmiller, M.Yurkov "Longitudinal Space Charge Driven Microbunching Instability in TTF2 Linac", *Proceedings of the ICFA Future Light Sources Sub-Panel Mini Workshop on Start-to-End Simulations of X-RAY FELs, Zeuthen* (2003), `http://www.desy.de/s2e/Talks/Monday/Talk_schneidmiller.pdf`

[2] M.Dohlus, "Space Charge Instability in XFEL", *talk given at the DESY XFEL Beam Dynamics Seminar* (21.11.2005), `http://www.desy.de/xfel-beam/data/talks/talks/→ →dohlus_-_sc_gain_500mev_20x5_20051121.pdf`

[3] M.Vogt, T.Limberg, D.Kuk, "Simulation of Micro Bunching Instability Regimes", THPC111, *EPAC2008, Genoa, Italy* (2008)

[4] Ph.Amstutz, M.Vogt, "A Time-Discrete Vlasov Approach to LSC Driven Microbunching in FEL-like Beamlines", *in this workshop* (2017)

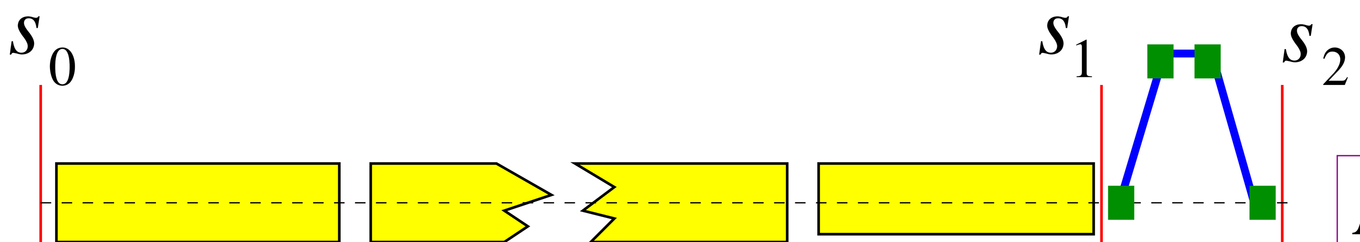
[5] R.L.Warnock, J.A.Ellison, "A General Method for Propagation of the Phase Space Distribution, with Application to the Sawtooth Instability", *Presented at The 2nd ICFA Advanced Accelerator Workshop on the Physics of High Brightness Beams, UCLA, 9-12/11/1999*, SLAC-PUB-8404 (03/2000)

[6] M.Vogt, J.A.Ellison, T.Sen, "Two Methods for Simulating the Strong–Strong Beam–Beam Interaction in Hadron Colliders", *Presented at Beam-Beam Workshop At Fermilab, 25-27/06/2001, Batavia, IL, USA*, SLAC-PUB-9576 (11/2002)

[7] J.Dieudonné, "Foundations of modern analysis", *Academic Press, Boston, MA, USA*, MR 0349288 (1969)

The Base Model

Single Bunch Compressor Stage



$$K[\Psi] = K_{\text{cav}} \circ K_{\text{lsc}} \; [\Psi]$$

$$D = D_{\text{chic}}$$

$$\boxed{M[\Psi] = D \circ K[\Psi]}$$

- Purely longitudinal selfconsistent model w/o CSR.

→ **longitudinal space charge** (LSC).

- Ultra–relativistic limit.
- Cartesian conjugate coordinates $q := -c\tau$ and $p := P_z - P_0 \approx E_z - E_0$. $\vec{z} := (q,p)^{\mathrm{T}}$
- Longitudinal phase space density (PSD) $\Psi(q,p)$.
- Longit. charge density of bunch with N particles $\rho(q) := eN \int_{\mathbb{R}} \rho(q,p)\,dp$.
- LSC ⇒ transfer map depends on density : $\vec{M} = \vec{M}[\Psi]$ (or $\vec{M}[\rho]$).
- All maps $\vec{K}$, $\vec{D}$, $\vec{M} \in \mathbf{Sp}(2,\mathbb{R})$

⇒ Vlasov evolution law for Ψ: $\boxed{\Psi_{\mathrm{f}} = \Psi_{\mathrm{i}} \circ M[\Psi_{\mathrm{i}}]^{-1}}$

- Model of **single bunch compressor stage** := { (long) **non–dispersive LinAcc + LSC** } followed by { (short) **magnetic chicane** }

→ $\vec{M}[\Psi] = \vec{D}^{\text{drift}}_{\text{magnetic chicane}} \circ \vec{K}^{\text{kick}}_{\text{cavities+LSC}}[\Psi]$.

- m bunch compressor stages : $\Psi_m = \Psi_{m-1} \circ \vec{M}_m[\Psi_{m-1}]^{-1}$.

→ **talk[4] by Ph. Amstutz!**

The (long) LinAcc/LSC Part $K[\Psi]$

- non–dispersive straight section

⇒ $\boxed{\rho(q) \text{ frozen!}}$

⇒ Transfer map is a **Poisson–type collective kick**

$$\begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} q \\ p + k_{\text{cav}}(q) + k_{\text{poi}}[\Psi](q) \end{pmatrix} ,$$

$$\Psi_{\mathrm{f}}(q,p) = \Psi_{\mathrm{i}}(q, p - k_{\text{cav}}(q) - k_{\text{poi}}[\Psi](q)) .$$

And since all kicks commute, kicks only change p, p integrated out of ρ, Poisson type kicks depend only on ρ:

$$k_{\text{poi}}[\Psi](q) := L_k \int_{\mathbb{R}^2} \langle G_{\text{poi}}\rangle \,(q,q')\Psi(q',p')\; dq'dp'$$

- k_{cav} : integral kick through the (chirped) acc. structures $k_{\text{cav}}(q) = \tilde{h}q + O(q^2)$, $\tilde{h} = R_{6,5}E_1$
- $\langle G_{\text{poi}}\rangle$: **kernel** for the LSC kick due to an axially symmetric beam with $\gamma(P_0)$ & transverse dimensions averaged over s,
- L_K = length of straight section (linac).

⇒ By modifying the energy density through the collective LSC–kick an *initial* current modulation can generate a *final* energy modulation!

The (short) Magnetic Chicane Part D

- Assume : Chicane short compared to LinAcc

→ neglect LSC.

- Moreover: ignore all synchrotron radiation effects (in particular CSR)

⇒ $\boxed{p\text{–density frozen!}}$

⇒ Transfer map is a (generalized) **Drift**

$$\begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} q + \lambda(p) \\ p \end{pmatrix} ,$$

$$\Psi_{\mathrm{f}}(q,p) = \Psi_{\mathrm{i}}(q - \lambda(p), p) ,$$

$$\lambda(p) := L_d\, p + O(p^2) \;,\; L_d := \frac{R_{5,6}}{P_0} .$$

- With (relative) E–chirp $h < 0$ ⇒ bunch compression with Factor $C := (1 + hR_{5,6})^{-1}$
- However, drift through chicane transforms an *initial* energy modulation into a *final* current modulation!

→ **Micro bunching : small initial modulations, amplified through successive bunch compressor stages.**

The Perron Frobenius Operator

- Let $\vec{M} : \mathbb{R}^2 \to \mathbb{R}^2$ be measure–preserving & invertible (e.g. symplectic),
- and $\Psi : \mathbb{R}^2 \to \mathbb{R}$ arbitrary in $\mathcal{L}^1(\mathbb{R}^2,\mathbb{R})$,
- then the composition $\Psi \circ \vec{M}^{-1}$ is in $\mathcal{L}^1(\mathbb{R}^2,\mathbb{R})$.

→ Perron Frobenius Operator[5, 6] (of $\vec{M}$) $\mathcal{M}$: $\mathcal{L}^1(\mathbb{R}^2,\mathbb{R}) \to \mathcal{L}^1(\mathbb{R}^2,\mathbb{R})$, $\Psi \mapsto \mathcal{M}\Psi := \Psi \circ \vec{M}^{-1}$ is a linear operator:

- $\mathcal{M}\,(\mu\Psi + \nu\Phi) = \mu\mathcal{M}\Psi + \nu\mathcal{M}\Phi$ for $\Psi, \Phi \in \mathcal{L}^1$ & $\mu, \nu \in \mathbb{R}$
- $\mathcal{M}\Psi$ describes the Liouville evolution of Ψ through $\vec{M}$. $\mathcal{M}[\Psi]\Psi$ describes the (selfconsistent) Vlasov evolution of Ψ through $\vec{M}$ which in turn depends (in a functional way) on Ψ

The Total (Frechet) Derivative

- Let $\|\cdot\|_{\text{op}}$ be a suitable operator norm on $\text{lin}(\mathcal{L}^1,\mathcal{L}^1)$ ($\mathcal{L}^1$ short for $\mathcal{L}^1(\mathbb{R}^2,\mathbb{R})$),
- the generalized total (Frechet) derivative[7] of $\mathcal{M}[\cdot] : \mathcal{L}^1 \to \text{lin}(\mathcal{L}^1,\mathcal{L}^1)$, $\Psi \mapsto \mathcal{M}[\Psi]$ at Ψ_0, is the linear operator $\mathcal{M}'[\Psi_0] \in \text{lin}(\mathcal{L}^1, \text{lin}(\mathcal{L}^1,\mathcal{L}^1))$ so that $\mathcal{M}[\Psi_0 + \phi] = \mathcal{M}[\Psi_0] + \mathcal{M}'[\Psi_0]\,\phi + o(\phi)$, equiv.:

$$\lim_{\phi\to 0} \frac{\|\mathcal{M}[\Psi_0+\phi] - \mathcal{M}[\Psi_0] - \mathcal{M}'[\Psi_0]\,\phi\|_{\text{op}}}{\|\phi\|_{\mathcal{L}^1}} = 0 \,,$$

given it exists.

The Linearized Vlasov Evolution for One BC Stage

- Given BC stage w/ map $\vec{M}[\Psi] = \vec{D} \circ \vec{K}_{\text{cav}} \circ \vec{K}_{\text{poi}}[\Psi] =: \vec{L} \circ \vec{K}_{\text{poi}}[\Psi]$,
- and a weakly smooth **reference** bunch PSD $\Psi_0 \in \mathcal{W}^1_1$ mapped by the BC stage to $\boxed{\Psi_1 := \mathcal{M}[\Psi_0]\Psi_0}$,
- the linearized Vlasov Evolution of the perturbation $\phi_0 \in \mathcal{L}^1$ around $(\Psi_0 \to \Psi_1)$ reads

$$\boxed{\begin{aligned} &\mathcal{M}[\Psi_0 + \varepsilon\phi_0]\,(\Psi_0 + \varepsilon\phi_0) = \\ &\mathcal{M}[\Psi_0]\,\Psi_0 + \varepsilon\left((\mathcal{M}'[\Psi_0]\cdot\phi_0)\;\Psi_0 + \mathcal{M}[\Psi_0]\,\phi_0\right) + O(\varepsilon^2) \\ &=: \Psi_1 + \varepsilon\phi_1 + O(\varepsilon^2) \,. \end{aligned}}$$

- $\phi_1 = \phi_0 \circ \vec{M}[\Psi_0]^{-1} - \partial_p\Psi_0 \circ \vec{M}[\Psi_0]^{-1} \cdot k_{\text{poi}}[\phi_0] \circ Q \circ \vec{L}^{-1}$, where $Q : (q,p)^{\mathrm{T}} \mapsto q$.

- or more explicitly

$$\boxed{\begin{aligned} \phi_1(\vec{z}) = &\;\phi_0\big(\tilde{Q}(\vec{z}), \tilde{P}(\vec{z}) - k_{\text{poi}}[\Psi_0](\tilde{Q}(\vec{z}))\big) \\ &-\partial_p\Psi_0\big(\tilde{Q}(\vec{z}), \tilde{P}(\vec{z}) - k_{\text{poi}}[\Psi_0](\tilde{Q}(\vec{z}))\big)\cdot \\ &\cdot k_{\text{poi}}[\phi_0](\tilde{Q}(\vec{z})) \,, \\ &\text{w/ } \tilde{Q}(\vec{z}) := (\vec{L}^{-1}(\vec{z}))_1 \;\&\; \tilde{P}(q,p) := (\vec{L}^{-1}(\vec{z}))_2 \end{aligned}}$$

- Let's now restrict to **linear** $\vec{D}$ and $\vec{K}_{\text{cav}}$, so that $\vec{L}(\vec{z}) = \underline{L}\vec{z}$,

w/ $\underline{L} \equiv \underline{D}\,\underline{K}_{\text{cav}} \equiv \begin{pmatrix} 1+L_d\tilde{h} & L_d \\ \tilde{h} & 1 \end{pmatrix}$.

$$\Rightarrow \boxed{\begin{aligned} \phi_1(q,p) = &\;\phi_0\big(\bar{Q}(q,p), \bar{P}(q,p) - k_{\text{poi}}[\Psi_0](\bar{Q}(q,p))\big) \\ &-\partial_p\Psi_0\big(\bar{Q}(q,p), \bar{P}(q,p) - k_{\text{poi}}[\Psi_0](\bar{Q}(q,p))\big)\cdot \\ &\cdot k_{\text{poi}}[\phi_0](\bar{Q}(q,p)) \,, \\ &\text{with } \bar{Q}(q,p) := (\underline{L}^{-1}\vec{z})_1 = q - L_d p, \\ &\text{and } \bar{P}(q,p) := (\underline{L}^{-1}\vec{z})_2 = -\tilde{h}q + (1 + L_d\tilde{h})p \end{aligned}}$$

n–th Order Pert. Expansion

- Now let $\Psi_0 \in \mathcal{W}^1_n$ and $\phi_0 \in \mathcal{W}^1_{n-1}$

$$\begin{aligned} \Rightarrow \mathcal{M}[\Psi_0 + \varepsilon\phi_0]\,(\Psi_0 + \varepsilon\phi_0) = \\ \Psi_1 + \textstyle\sum_{k=1}^n \varepsilon^k \cdot \\ \cdot\Big(\frac{(-\partial_p)^{k-1}}{(k-1)!}\phi_0 \circ \vec{M}[\Psi_0]^{-1} \cdot (k_{\text{poi}}[\phi_0] \circ \tilde{Q})^{k-1} \\ \frac{(-\partial_p)^k}{k!} \;\; \Psi_0 \circ \vec{M}[\Psi_0]^{-1} \cdot (k_{\text{poi}}[\phi_0] \circ \tilde{Q})^k \quad \Big) \\ +O(\varepsilon^{n+1}) \\ =: \Psi_1 + \textstyle\sum_{k=1}^n \varepsilon^k\phi_{1,k} + O(\varepsilon^{n+1}) \end{aligned}$$

- i.p. $\phi_{1,k} \equiv \phi_1$ (as above), and
- $\phi_{1,2}(\vec{z}) =$

$$\begin{aligned} &-\partial^1_p\phi_0\big(\tilde{Q}(\vec{z}), \tilde{P}(\vec{z}) - k_{\text{poi}}[\Psi_0](\tilde{Q}(\vec{z}))\big) \cdot \big(k_{\text{poi}}[\phi_0](\tilde{Q}(\vec{z}))\big)^1 \\ &+\tfrac{\partial^2_p}{2}\Psi_0\big(\tilde{Q}(\vec{z}), \tilde{P}(\vec{z}) - k_{\text{poi}}[\Psi_0](\tilde{Q}(\vec{z}))\big) \cdot \big(k_{\text{poi}}[\phi_0](\tilde{Q}(\vec{z}))\big)^2 \end{aligned}$$

- Note that the "self modulation" term of ϕ_0: $\partial_p\phi_0 \circ \vec{M}[\Psi_0]^{-1} \cdot k_{\text{poi}}[\phi_0] \circ \tilde{Q}$ only enters at 2nd order!

Cascade of m BC Stages

- We slightly change our notation here:

for stage l **:** $\Psi_0 \to \psi_{l,0}$, $\phi_{0,k} \to \psi_{l,k}$, $1 \le k \le n$,
$\vec{D}, \vec{K}, \vec{L}, \vec{M}, \tilde{Q}, \tilde{P}, k_{\text{poi}}, \ldots \;\to\; \vec{D}_l, \vec{K}_l, \vec{L}_l, \vec{M}_l, \tilde{Q}_l, \tilde{P}_l, k_l, \ldots$
mapping from stage l **to** $l+1$.

$$\begin{aligned} \Rightarrow &\textstyle\sum_{k=0}^n \varepsilon^k\psi_{l+1,k} + O(\varepsilon^{n+1}) = \mathcal{M}_l[\sum_{k=0}^n \varepsilon^k\psi_{l,k}]\; \sum_{k=0}^n \varepsilon^k\psi_{l,k} =_n \\ &\textstyle\sum_{k=0}^n \varepsilon^k \Big(\psi_{l,k}\big(\tilde{Q}_l, \tilde{P}_l - k_l[\psi_{l,0}](\tilde{Q}_l) - \sum_{j=1}^{n-k}\varepsilon^k k_l[\psi_{l,k}](\tilde{Q}_l)\big)\Big) =_n \\ &\textstyle\sum_{k=0}^n \varepsilon^k \Big(\sum_{j=0}^{n-k} \frac{(-\partial_p)^j}{j!}\psi_{l,k}\big(\tilde{Q}_l, \tilde{P}_l - k_l[\psi_{l,0}](\tilde{Q}_l)\big)\cdot \\ &\qquad \cdot\big(\textstyle\sum_{i=1}^{n-k} \varepsilon^i k_l[\psi_{l,i}](\tilde{Q}_l)\big)^{j\star(n-k)} \quad \Big) \end{aligned}$$

- where $a =_n b$ is short for $a = b + O(\varepsilon^{n+1})$,

- and $(\sum_{j=1}^{n-k} \varepsilon^k a_k)^{j\star(n-k)}$ is $(\sum_{j=1}^{n-k} \varepsilon^k a_k)^j$ w/ terms only up to order ε^{n-k}.
- i.p. for $n = 1$ and $m = 2$ we find $\psi_{2,0} = \mathcal{M}_1[\psi_{1,0}]\psi_{1,0} = \mathcal{M}_1[\mathcal{M}_0[\psi_{0,0}]\psi_{0,0}]\;\mathcal{M}_0[\psi_{0,0}]\;\psi_{0,0}$,

and $\psi_{2,1} =$

$$\begin{aligned} &\psi_{1,1}\big(\tilde{Q}_1, \tilde{P}_1 - k_1[\psi_{1,0}](\tilde{Q}_1)\big) \\ &-\partial_p\psi_{1,0}\big(\tilde{Q}_1, \tilde{P}_1 - k_1[\psi_{1,0}](\tilde{Q}_1)\big) \cdot k_1[\psi_{1,1}](\tilde{Q}_1) \end{aligned}$$

where $\psi_{1,0} = \mathcal{M}_0[\psi_{0,0}]\;\psi_{0,0}$ and $\psi_{1,1} =$

$$\begin{aligned} &\psi_{0,1}\big(\tilde{Q}_0, \tilde{P}_0 - k_0[\psi_{0,0}](\tilde{Q}_0)\big) \\ &-\partial_p\psi_{0,0}\big(\tilde{Q}_0, \tilde{P}_0 - k_0[\psi_{0,0}](\tilde{Q}_0)\big) \cdot k_0[\psi_{0,1}](\tilde{Q}_0) \end{aligned}$$

- higher orders and deeper iterations are possible but too lengthy (and unpleasant) to present on this poster :-)

Gain Functions

- starting from an initial perturbation $\phi_0 \equiv \psi_{0,1}(q,p)$ with projected spacial density $\rho_{0,1}(q)$ and Fourier transform $\hat{\rho}_{0,1}(\kappa)$,
- one may define the most general n-th order m stage gain function: $g^{(n,m)}[\Psi_0, \vec{M}_0, \cdots, \vec{M}_m; \phi_0](\kappa_{\mathrm{i}}, \kappa_{\mathrm{f}}) := \frac{\hat{\rho}_{m,n}(\kappa_{\mathrm{f}})}{\hat{\rho}_{0,1}(\kappa_{\mathrm{i}})}$ and the accumulated general n-th order m stage gain function: $\Gamma^{(n,m)}[\ldots](\kappa_{\mathrm{i}}, \kappa_{\mathrm{f}}) := \sum_{k=1}^n g^{(k,m)}[\ldots](\kappa_{\mathrm{i}}, \kappa_{\mathrm{f}})$
- if the **chirp** of the reference PSDs $(\Psi_0, \ldots, \Psi_{m-1})$ does only **vary weakly** over the bunchlength, and if generation of harmonics is neglected, it is often convenient to define the **compression corrected**, absolute gain: $\tilde{g}^{(n,m)}[\ldots](\kappa) := |g^{(n,m)}[\ldots](\kappa, \kappa\cdot(C_0\cdots C_m))|$ and $\tilde{\Gamma}^{(n,m)}[\ldots](\kappa) := |\Gamma^{(n,m)}[\ldots](\kappa, \kappa\cdot(C_0\cdots C_m))|$

Example: Infinitely Long Bunch

- $\Psi_0(q,p) \equiv \psi_{0,0}(q,p) = \xi_{\mu_0(q),\sigma_0}(p)\Lambda_{t_0}(q)$, $\xi_{\mu,\sigma}$:= Gaussian w/ parm's μ, σ, $\mu_0(q) \equiv 0$, $\sigma_0 = \text{const.} > 0$; Λ_{t_0} is constant $\sim T_0 := 1/(2t_0)$ over $[-t_0, t_0]$; assumed characteristic *transverse* size is a and $t_0 \gg \sigma_0, a$. Then $\rho_{0,0}(q) = \Lambda_h(q)$ and furthermore $k_{\text{poi}}[\Psi_0] \approx 0$ deep enough *inside* the bunch and for any decent $\langle G_{\text{poi}}\rangle$. We neglect all edge effects here!
- $\vec{D}$, $\vec{K}_{\text{cav}}$ linear, $L_d > 0$, $\tilde{h} < 0$

⇒ $\underline{L}$ =naive bunch compressor w/ compression $C > 1$

- $\Psi_1(q,p) \equiv \psi_{1,0}(q,p) \approx \Psi_0(\bar{Q}(q,p), \bar{P}(q,p))$, i.p. $\mu_1(q) = \tilde{h}q$, $t_1 \approx t_0/C$, $T_1 = CT_0$
- Now assume $\phi_0(q,p) \equiv \psi_{0,1}(q,p) = \eta\psi_{0,0}(q,p)\cos(\kappa_{\mathrm{i}}q)$, w/ $0 \le \eta < 1$, so that $\Psi_0 + \phi_0 \ge 0$ and $\int_{\mathbb{R}^2}(\Psi_0+\phi_0)\,dqdp \approx 1$ for $1/k_i \ll t_0/C$. ⇒ $\rho_{0,1}(q) = \eta\Lambda_h(q)\cos(\kappa_{\mathrm{i}}q)$
- $\hat{G}_{\text{poi}}(\kappa) \propto \frac{a^2}{\kappa^2}\big(1 - 2I_1(|\kappa|a)K_1(|\kappa|a)\big)$ (mean force on a–disk due to a charged a–disk)

⇒ $k_{\text{poi}}[\phi_0](q) \propto \eta\hat{G}_{\text{poi}}(\kappa_{\mathrm{i}})\cos(\kappa_{\mathrm{i}}q)$ (deep inside bunch)

- choose $n = 1$ & $m = 2$ ⇒

$$\begin{aligned} \Rightarrow \psi_{1,1} \approx \;\; &\phi_0(\bar{Q}, \bar{P}) \\ &-\partial_p\Psi_0(\bar{Q}, \bar{P}) \cdot \eta\alpha_{\text{poi}}\hat{G}_{\text{poi}}(\kappa_{\mathrm{i}})\cos(\kappa_{\mathrm{i}}\bar{Q}) \end{aligned}$$

$$\begin{aligned} \Rightarrow \psi_{1,2} \approx \;\; &\partial_p\phi_0(\bar{Q}, \bar{P}) \cdot \eta\alpha_{\text{poi}}\hat{G}_{\text{poi}}(\kappa_{\mathrm{i}})\cos(\kappa_{\mathrm{i}}\bar{Q}) \\ &-\tfrac{\partial^2_p}{2}\Psi_0(\bar{Q}, \bar{P}) \cdot \big(\eta\alpha_{\text{poi}}\hat{G}_{\text{poi}}(\kappa_{\mathrm{i}})\cos(\kappa_{\mathrm{i}}\bar{Q})\big)^2 \end{aligned}$$

- Note that since $\xi'_{\mu,\sigma}(p) = \frac{p-\mu}{\sigma^2}\,\xi_{\mu,\sigma}(p)$ and $\xi''_{\mu,\sigma}(p) = \big(\frac{(p-\mu)^2}{\sigma^4} - \frac{1}{\sigma^2}\big)\,\xi_{\mu,\sigma}(p)$, all functions above can be evaluated explicitly.
- Now let the chirp $\tilde{h}$ be so weak, that the compression $C = 1/(1 + \tilde{h}L_d) \approx 1$, despite finite L_d. Then $\bar{P}(q,p) \approx p$, and since $\Lambda_{t_0}(q)$ is constant well inside the bunch, the projections $\psi \circ \vec{L} \to \rho$ can be performed w/ $\hat{G}_{\mathrm{i}} := \alpha_{\text{poi}}\hat{G}_{\text{poi}}(\kappa_{\mathrm{i}})$:

$$\begin{aligned} \to \rho_{1,1} \approx \;\; &\eta\Lambda_{t_0}\cos(\kappa_{\mathrm{i}}q) \\ &-\tfrac{\eta\Lambda_{t_0}\hat{G}_{\mathrm{i}}}{\sigma^2}\sin(\kappa_{\mathrm{i}}q)\,2\textstyle\int_0^\infty p\,\xi_{0,\sigma}(p)\sin(\kappa_{\mathrm{i}}L_d p)\,dp \end{aligned}$$

since $\xi'_{0,\sigma}(p)$ is odd

$$\begin{aligned} \to \rho_{1,2} \approx \;\; &\tfrac{\eta^2\Lambda_{t_0}\hat{G}_{\mathrm{i}}}{2\sigma^2}\sin(2\kappa_{\mathrm{i}}q)\,2\textstyle\int_0^\infty p\,\xi_{0,\sigma}(p)\sin(2\kappa_{\mathrm{i}}L_d p)\,dp \\ &-\tfrac{\eta^2\Lambda_{t_0}\hat{G}_{\mathrm{i}}}{4\sigma^2}\cos(2\kappa_{\mathrm{i}}q)\,2\textstyle\int_0^\infty (\frac{p^2}{\sigma^2} - 1)\,\xi_{0,\sigma}(p)\cos(2\kappa_{\mathrm{i}}L_d p)\,dp \end{aligned}$$

since $\xi''_{0,\sigma}(p)$ is even

- The gain functions are easily computed since the only q–dependence (neglecting the edge effects) is in the sin-, cos–terms of argument $\kappa_{\mathrm{i}}q$, $2\kappa_{\mathrm{i}}q$.

General Remarks

- No explicit knowledge on Ψ_1 is needed for single stage system!
- For $\partial^n_p\Psi$ to exist (at least in the weak sense) $\Psi \in \mathcal{L}^1$ is most likely not enough:

→ instead $\Psi \in \mathcal{W}^1_n$ (Sobolev-n-1) or similar is necessary.

→ i.p. the model is not very likely to ab applicable to strongly "curled up" reference bunches as they sometimes appear in FELs.

⇒ numerics needed → Ph. Amstutz's talk[4]!

- Rigorous error bounds have not yet been studied.
- Even for smooth reference bunches, the compression corrected gain does not capture fancy nonlinear features like generation of harmonics, e.t.c.
- This is work in progress, still more to explore!

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